Petersson inner products of weight one modular forms

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Abstract

In this paper we study regularized Petersson products between a holomorphic theta series associated to a positive definite binary quadratic form and a weakly holomorphic weight 1 modular form with integral Fourier coefficients. In our recent work [17] motivated by the conjecture of B. Gross and D. Zagier on the CM values of higher Green's functions we have discovered that such a Petersson product is equal to the logarithm of a certain algebraic number lying in a ring class field associated to the binary quadratic form. The main result of the present paper is the explicit factorization formula for the obtained algebraic number.

1 Introduction

In this paper we study arithmetic properties of the regularized Petersson product between the following two modular functions of weight one: holomorphic binary theta series and a weakly holomorphic modular form with integral Fourier coefficients.

More precisely, consider an imaginary quadratic field $K := \mathbb{Q}(\sqrt{-D})$. For simplicity we assume that D is a prime congruent to 3 modulo 4. Let $\mathfrak{b}$ be an element of ideal class group of K. Denote by $r_{\mathfrak{b}}(t)$ the number of integral ideals of norm t in the ideal class $\mathfrak{b}$. The theta function

$$\Theta_{\mathfrak{b}}(\tau) = \sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} e_\nu \sum_{t\equiv\nu^2 \bmod D} (1+\delta_{0,\nu})\, r_{\mathfrak{b}}(t)\, \mathbf{e}\Big(\frac{t}{D}\tau\Big) \tag{1}$$

is a holomorphic vector valued modular form of weight 1 and a representation ρ defined in Section 3. Here we use the standard notation $\mathbf{e}(x) := e^{2\pi i x}$.

First, consider the classical Petersson inner product between $\Theta_{\mathfrak{b}}$ and the cusp form $g_\chi := \sum_{\mathfrak{c}\in \mathrm{CL}_K} \chi(\mathfrak{c})\Theta_{\mathfrak{c}}$ associated to a character $\chi : \mathrm{CL}_K \to \mathbb{C}^\times$

$$(g_\chi, \Theta_{\mathfrak{b}}) := \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \langle g_\chi(\tau), \overline{\Theta_{\mathfrak{b}}(\tau)}\rangle \, y^{-1}\, dx\, dy,$$

where $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ and $\tau = x + iy$. Applying the Rankin-Selberg method to this integral one can see that

$$(g_\chi, \Theta_{\mathfrak{b}}) = \frac{1}{h}\chi(\mathfrak{b}^{-1})\, L(\chi_D, 1)\, \mathrm{res}_{s=1} L_K(\chi^2, s), \tag{2}$$

where h is the class number of K and $\chi_D(\cdot) = \left(\frac{\cdot}{D}\right)$. Thus, the Stark's theorem implies

$$(g_\chi, \Theta_{\mathfrak{b}}) = \sum_{\mathfrak{c}\in \mathrm{CL(K)}} \chi^2(\mathfrak{c}) \log|\varepsilon_{\mathfrak{c}}| \tag{3}$$

for certain units $\varepsilon_{\mathfrak{c}}$ in the Hilbert class field of K.

The theory of regularized theta lift developed by Borcherds, Bruinier, Kudla and others motivates us to generalize the classical identity (3) by replacing the cusp form g_χ with a weakly holomorphic modular form. More precisely, we call f a weakly holomorphic cusp form if it is a weakly holomorphic (vector valued) modular form and has zero constant term. We denote by $S_1^!(\rho)$ the space of weakly holomorphic cusp form of weight 1 and representation ρ. For $f \in S_1^!(\rho)$ we define a regularized Petersson product as

$$(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} := \lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle f(\tau), \overline{\Theta_{\mathfrak{b}}(\tau)}\rangle \, y^{-1}\, dx\, dy, \tag{4}$$

where

$$\mathcal{F}_T = \left\{\tau \in \mathfrak{H} \middle| -1/2 < \Re(\tau) < 1/2,\ |\tau| > 1, \text{ and } \Im(\tau) < T\right\}$$

is the truncated fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. We are interested in the arithmetic properties of the number (4) when f has integral fourier coefficients. In our recent work [17] inspired by the conjecture of B. Gross and D. Zagier on the CM values of higher Green's functions we have discovered that

$$(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} = \log|\alpha| \tag{5}$$

for some algebraic number α lying in an abelian extension of K. Similar result was obtained independently by W. Duke and Y. Li [8], however their interest in this problem arose from the theory of mock modular forms. The main result of the present paper is the explicit factorization formula for the algebraic number α in (5).

Let h be the class number and H be the Hilbert class field of K, respectively. For an ideal class $\mathfrak{c} \in \mathrm{CL}(K)$ we denote by $\sigma_{\mathfrak{c}}$ the element of $\mathrm{Gal}(H/K)$ corresponding to $\mathfrak{c}$ under the Artin isomorphism. Fix an embedding $\imath : H \to \mathbb{C}$. Let p be a rational prime with $\left(\frac{p}{D}\right) = -1$. Let $\mathcal{P}_p = \{\wp_i\}_{i=0}^{h}$ be the set of prime ideals of H lying above p. The complex conjugation acts on this set. Since the class number h is odd, there exist a unique prime ideal in $\mathcal{P}_p$, say $\wp_1$, with $\wp_1 = \overline{\wp_1}$. For a prime ideal $\wp \in \mathcal{P}_p$ there exists a unique element $\sigma \in \mathrm{Gal}(H/K)$ such that

$$\wp^{\sigma} = \wp_1. \tag{6}$$

Denote by $\mathfrak{a} = \mathfrak{a}(\wp)$ a fractional ideal in K whose class corresponds to σ under the Artin isomorphism.

Theorem 1. *Let $\mathfrak{b} \in$ be an ideal class of K and let $f \in S_1^{!}(\rho)$ be a weakly holomorphic modular form with the Fourier expansion*

$$f = \sum_{\nu \in N'/N} e_{\nu} \sum_{\substack{t \in \mathbb{Z} \\ t \gg -\infty}} c_{\nu}(t)\, \mathbf{e}\Big(\frac{t}{D}\tau\Big)$$

satisfying $c_0(0) = 0$ and $c_{\nu}(m) \in \mathbb{Z}$ for all $\nu \in \mathbb{Z}/D\mathbb{Z}$. Then there exists an algebraic number $\alpha \in H$ such that

$$(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} = \log|\alpha|.$$

Moreover, for a rational prime p with $\left(\dfrac{p}{D}\right) = -1$ and a prime $\wp$ lying above p in the Hilbert class field H we have

$$\mathrm{ord}_{\wp}(\alpha) = \sum_{t<0} \sum_{\nu \in \mathbb{Z}/D\mathbb{Z}} c_{\nu}(t)\, r_{\mathfrak{b}\mathfrak{a}^2}\Big(\frac{-t}{p}\Big)\, \mathrm{ord}_p(t). \tag{7}$$

Here $r_{\mathfrak{b}\mathfrak{a}^2}$ is defined as in (1).

Remark 1. Theorem 1 is compatible with, but stronger then the result of J. Schofer [16]. More precisely, Theorem 4.1 on p.30 of [16] states that the *sum* over all isomorphism classes of even lattices of discriminant D of the identity (1) is true.

Remark 2. Theorem 1 is stated in a conjectural form in [8].

Remark 3. For simplicity we don't consider the primes p with $\left(\frac{p}{D}\right)$ equal to 0 or 1.

Remark 4. The space of vector valued modular forms $M_1(\rho)$ is isomorphic to the space $M_1^+(\Gamma_0(D), \chi_D)$ of scalar valued modular forms, see [5].

The idea of the proof is to use the embedding argument of Theorem 5 in [17] and reduce the computation of the regularized integral $(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}}$ to the computation of the local height pairings between certain Heegner points on the modular curve $X_0(D)$.

The paper is organized as follows. In Section 2 we recall the definition and basic properties of vector valued modular forms. In Sections 3 and 4 we collect necessary facts from the theory of Borcherds lift. Also we give a brief review of height theory on the curves in Section 5. In Section 6 we construct a certain meromorphic function Ψ on the modular curve $X_0(D)$. This function has zeroes and poles at the Heegner points and satisfies $(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} = \log|\Psi(\mathfrak{z})|$ for some CM-point $\mathfrak{z}$. In Section 7 we use the computations of the local height pairing made by B.Gross and D. Zagier in [12] and find the local pairings between $\mathfrak{z}$ and $\mathrm{div}(\Psi)$ over the finite places of H. This gives as the valuation of $\alpha = \Psi(\mathfrak{z})$ at the primes of H and finishes the proof of Theorem 1.

2 Lattices and vector valued modular forms

Recall that the group $\mathrm{SL}_2(\mathbb{Z})$ has a double cover $\mathrm{Mp}_2(\mathbb{Z})$ called the *metaplectic group* whose elements can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d}\right)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\sqrt{c\tau + d}$ is considered as a holomorphic function of τ in the upper half plane whose square is $c\tau + d$. The multiplication

is defined so that the usual formulas for the transformation of modular forms of half integral weight work, which means that

$$(A, f(\tau))(B, g(\tau)) = (AB, f(B(\tau))g(\tau))$$

for $A, B \in \mathrm{SL}_2(\mathbb{Z})$ and f, g suitable functions on $\mathfrak{H}$.

Let (V, q) be a rational quadratic space over $\mathbb{Q}$, that is a rational vector space V equipped with the quadratic form $\mathrm{q} : V \to \mathbb{Q}$. The corresponding bilinear form on $V \times V$ is defined by $(x, y) = \frac{1}{2}\mathrm{q}(x+y) - \frac{1}{2}\mathrm{q}(x-y)$. Suppose that V has signature (b^+, b^-). Let $L \subset V$ be a lattice. The dual lattice of L is defined as $L' = \{x \in V | (x, L) \subset \mathbb{Z}\}$. We say that L is even if $\mathrm{q}(\ell) \in \mathbb{Z}$ for all $\ell \in L$. In this case L is contained in L' and L'/L is a finite abelian group.

We let the elements e_ν for $\nu \in L'/L$ be the standard basis of the group ring $\mathbb{C}[L'/L]$, so that $e_\mu e_\nu = e_{\mu+\nu}$. The complex conjugation acts on $\mathbb{C}[L'/L]$ by $\overline{e_\mu} = e_\mu$. Consider the scalar product on $\mathbb{C}[L'/L]$ given by

$$\langle e_\mu, e_\nu \rangle = \delta_{\mu,\nu} \tag{8}$$

and extended to $\mathbb{C}[L'/L]$ by linearity. Recall that there is a unitary representation ρ_L of the double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(\widetilde{T})(e_\nu) = \mathbf{e}\big(\mathrm{q}(\nu)\big)\, e_\nu \tag{9}$$

$$\rho_L(\widetilde{S})(e_\nu) = i^{(b^-/2 - b^+/2)}\, |L'/L|^{-1/2} \sum_{\mu \in L'/L} \mathbf{e}\big(-(\mu, \nu)\big) e_\mu, \tag{10}$$

where

$$\widetilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \text{and } \widetilde{S} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \tag{11}$$

are the standard generators of $\mathrm{Mp}_2(\mathbb{Z})$.

For an integer $n \in \mathbb{Z}$ we denote by $L(n)$ the lattice L equipped with a quadratic form $q^{(n)}(\ell) := n\mathrm{q}(\ell)$. In the case $n = -1$ the lattices $L'(-1)$ and $(L(-1))'$ coincide and hence the groups L'/L and $L(-1)'/L(-1)$ are equal. Both representations ρ_L and $\rho_{L(-1)}$ act on $\mathbb{C}[L'/L]$ and for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ we have $\rho_{L(-1)}(\gamma) = \overline{\rho_L(\gamma)}$.

A vector valued modular form of half-integral weight k and representation ρ_L is a function $f : \mathfrak{H} \to \mathbb{C}[L'/L]$ that satisfies the following transformation law

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\sqrt{c\tau+d}^{2k}\rho_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix},\sqrt{c\tau+d}\right)f(\tau).$$

We will use the notation $\mathfrak{M}_k(\rho_L)$ for the space of real analytic, $M_k(\rho_L)$ for the space of holomorphic, and $M^!_k(\rho_L)$ for the space of weakly holomorphic modular forms of weight k and representation ρ_L. We denote by $S^!_k(\rho_L)$ the space of weakly holomorphic modular forms of weight k and representation ρ_L with zero constant term.

Now we recall some standard maps between the spaces of vector valued modular forms of associated to different lattices [7].

If $M \subset L$ is a sublattice of finite index then a vector valued modular form $f \in \mathfrak{M}_k(\rho_L)$ can be naturally viewed as a vector valued modular form in $f \in \mathfrak{M}_k(\rho_M)$. Indeed, we have the inclusions

$$M \subset L \subset L' \subset M'$$

and therefore

$$L/M \subset L'/M \subset M'/M.$$

We have the natural map $L'/M \to L'/L$, $\mu \to \bar{\mu}$.

Lemma 1. *For $\mathcal{M}=\mathfrak{M}, M$ or $M^!$ there are two natural maps*

$$\mathrm{res}_{L/M}: \mathcal{M}_k(\rho_L) \to \mathcal{M}_k(\rho_M),$$

and

$$\mathrm{tr}_{L/M}: \mathcal{M}_k(\rho_M) \to \mathcal{M}_k,(\rho_L),$$

given by

(12)
$$\big(\mathrm{res}_{L/M}(f)\big)_\mu = \begin{cases} f_{\bar{\mu}}, & \textit{if } \mu \in L'/M \\ 0 & \textit{if } \mu \notin L'/M \end{cases}, \qquad \big(f \in \mathcal{M}_k(\rho_L),\ \mu \in M'/M\big)$$

and

(13)
$$\big(\mathrm{tr}_{L/M}(g)\big)_\lambda = \sum_{\mu \in L'/M\colon \bar{\mu}=\lambda} g_\mu, \qquad \big(g \in \mathcal{M}_k(\rho_M),\ \ \lambda \in L'/L\big).$$

Now suppose that M and N are two even lattices and $L = M \oplus N$. Then we have

$$L'/L \cong (M'/M) \oplus (N'/N).$$

Moreover

$$\mathbb{C}[L'/L] \cong \mathbb{C}[M'/M] \otimes \mathbb{C}[N'/N]$$

as unitary vector spaces and naturally

$$\rho_L = \rho_M \otimes \rho_N.$$

Lemma 2. *For two modular forms $f \in \mathcal{M}_k(\rho_L)$ and $g \in \mathcal{M}_l(\rho_{M(-1)})$ the function*

$$h := \langle f, g\rangle_{\mathbb{C}[M'/M]} = \sum_{\nu \in N'/N} e_\nu \sum_{\mu \in M'/M} f_{\mu\oplus\nu}\, g_\mu$$

belongs to $\mathcal{M}_{k+l}(\rho_N)$.

3 Theta functions and Theta lifts

In this section we recall the definition of regularized theta lift given by Borcherds in the paper [2].

We let L be an even lattice of signature $(2, b^-)$ with dual L'. The (positive) Grassmannian $\mathrm{Gr}^+(L)$ is the set of positive definite two dimensional subspaces v^+ of $L \otimes \mathbb{R}$. We write v^- for the orthogonal complement of v^+, so that $L \otimes \mathbb{R}$ is the orthogonal direct sum of the positive definite subspace v^+ and the negative definite subspace v^-. The projection of a vector $\ell \in L \otimes \mathbb{R}$ into a subspaces v^+ and v^- is denoted by ℓ_{v^+} and ℓ_{v^-} respectively, so that $\ell = \ell_{v^+} + \ell_{v^-}$.

The vector valued Siegel theta function $\Theta_L : \mathfrak{H} \times \mathrm{Gr}^+(L) \to \mathbb{C}[L'/L]$ of L is defined by

$$\Theta_L(\tau, v^+) = y^{b^-/2} \sum_{\ell \in L'} \mathbf{e}\big(\mathrm{q}(\ell_{v^+})\tau + \mathrm{q}(\ell_{v^-})\bar{\tau}\big)\, e_{\ell+L}. \tag{14}$$

Theorem 4.1 in [2] says that $\Theta_L(\tau, v^+)$ is a real-analytic vector valued modular form of weight $1 - b^-/2$ and representation ρ_L with respect to variable τ.

For $f \in \mathfrak{M}_{1-b/2}(\rho_L)$ we define a *regularized theta integral* as

$$\Phi_L(v^+, f) := \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} \langle f(\tau), \overline{\Theta_L(\tau, v^+)}\rangle\, y^{-1-b^-/2}\, dx\, dy \tag{15}$$

(the product of Θ_L and f means we take their inner product using $\langle e_\mu, e_\nu\rangle = 1$ if $\mu = \nu$ and 0 otherwise.)

The integral is often divergent and has to be regularized. In this paper we consider regularized lifts of weakly holomorphic cusp forms. In this case the regularization is simpler than in the general situation. For $f \in S^!_{1-b/2}(\rho_L)$ we set

$$\Phi_L(v^+, f) := \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \overline{\Theta_L(\tau, v^+)}\rangle\, y^{-1-b^-/2}\, dx\, dy,$$

where $\mathcal{F}_T$ is a truncated fundamental domain introduced in Section 1.

Denote by $\mathrm{Aut}(L)$ the group of those isometries of $L \otimes \mathbb{R}$ that fix L. The action of $\mathrm{Aut}(L)$ on f is given by action on L'/L. We define $\mathrm{Aut}(L, f)$ to be the subgroup of $\mathrm{Aut}(L)$ fixing f. The regularized integral $\Phi_L(v^+, f)$ is a function on the Grassmannian $\mathrm{Gr}^+(L)$ that is invariant under $\mathrm{Aut}(L, f)$.

In the case when L has signature $(2, b^-)$ the Grassmanian $\mathrm{Gr}^+(L)$ carries the structure of a Hermitian symmetric space. If X and Y are an oriented orthogonal base of some element v^+ of $\mathrm{Gr}(V)$ then we map v^+ to the point of the complex projective space $\mathbb{P}(V \otimes \mathbb{C})$ represented by $Z = X + iY \in V \otimes \mathbb{C}$. The fact that $Z = X + iY$ has norm 0 is equivalent to saying that X and Y are orthogonal and have the same norm. This identifies $\mathrm{Gr}^+(V)$ with an open subset of the norm 0 vectors of $\mathbb{P}(V \otimes \mathbb{C})$ in a canonical way, and gives $\mathrm{Gr}^+(V)$ a complex structure invariant under the subgroup $\mathrm{O}^+(V\otimes\mathbb{R})$ of index 2 of $\mathrm{O}(V \otimes \mathbb{R})$ of elements preserving the orientation on the 2 dimensional positive definite subspaces. Thus, the open subset

$$\mathcal{P} = \left\{[Z] \in \mathbb{P}(V \otimes \mathbb{C}) \middle| (Z, Z) = 0 \text{ and } (Z, \overline{Z}) > 0\right\}$$

is isomorphic to $\mathrm{Gr}^+(V)$ by mapping $[Z]$ to the subspace $\mathbb{R}\Re(Z) + \mathbb{R}\Im(Z)$.

The following theorem of Borcherds relates regularized theta lifts associated to the lattice L of signature $(2, b)$ with infinite products introduced in his earlier paper [1].

Theorem 2. *([2], Theorem 13.3) Suppose that $f \in S^!_{1-b/2}(\rho_L)$ has the Fourier expansion*

$$f(\tau) = \sum_{\lambda\in L'/L} \sum_{n\gg-\infty} c_\lambda(n)\, \mathbf{e}(n\tau)\, e_\lambda$$

and the Fourier coefficients $c_\lambda(n)$ are integers for $n \le 0$. Then there is a meromorphic function $\Psi_L(Z, f)$ on $\mathcal{P}$ with the following properties.

1. *Ψ is an automorphic function for the group* $\mathrm{Aut}(L, f)$ *with respect to some unitary character of* $\mathrm{Aut}(L, f)$

2. *The only zeros and poles of Ψ_L lie on the rational quadratic divisors $\ell^\perp$ for $\ell \in L$, $\mathrm{q}(\ell) < 0$ and are zeros of order*

$$\sum_{\substack{x \in \mathbb{R}^+: \\ xl \in L'}} c_{xl}\big(\mathrm{q}(xl)\big)$$

3. $$\Phi_L(Z, f) = -4 \log |\Psi_L(Z, f)|.$$

4. *One can write an explicit infinite product expansion converging in a neighborhood of each cusp of* $\mathrm{Gr}^+(L)$.

Remark 5. Theorem 13.3 in [2] is formulated in more general settings and an explicit infinite product expansion is given there.

At the end of this section let us consider the lattices of signature $(2, 0)$ in more detail. Recall that there is a one-to-one correspondence between equivalence classes of even lattices of fundamental discriminant $-D$ and fractional ideals of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. For a fractional ideal $\mathfrak{b} \subset K$ consider an even lattice $N = (\mathfrak{b}, \mathrm{q})$, where the quadratic form q is defined as $\mathrm{q}(x) = \frac{1}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b})} \mathrm{N}_{K/\mathbb{Q}}(x)$ for $x \in \mathfrak{b}$. The isomorphism class of N depends only on the ideal class of $\mathfrak{b}$. Moreover, the representation ρ_N defined in Section 2 depends only on the genus of N. Thus, for D prime the representation ρ_N is the same for all fractional ideals of K and we denote this representation by ρ. We note that the theta function $\Theta_{\mathfrak{b}}$ defined in Section 1 coincides with a theta function Θ_N defined by (14). It follows from the definition of theta lift that for $f \in S_1^!(\rho)$

$$\Phi_N(f) = (f, \Theta_{\mathfrak{b}})^{\mathrm{reg}}.$$

Thus, the regularized peterson product (4) can be seen as a theta lift of f to a zero dimensional Grassmanian $\mathrm{Gr}^+(N)$.

4 A see-saw identity

In the paper [14] S. Kudla introduced the notion of a “see-saw dual reductive pair”. It gives rise in a formal way to a family of identities between inner

products of automorphic forms on different groups, thus clarifying the source of identities of this type which appear in many places in the literature, often obtained from complicated manipulations. Here we prove a see-saw identity for the regularized theta integrals described.

Suppose that (V, q) is a rational quadratic space of signature $(2, b)$ and $L \subset V$ is an even lattice. Let $V = V_1 \oplus V_2$ be the rational orthogonal splitting of (V, q) such that the space V_1 has the signature $(2, b-d)$ and the space V_2 has the signature $(0, d)$. Consider two lattices $N := L \cap V_1$ and $M := L \cap V_2$. We have two orthogonal projections

$$\mathrm{pr}_M : L \otimes \mathbb{R} \to M \otimes \mathbb{R} \quad \text{and} \quad \mathrm{pr}_N : L \otimes \mathbb{R} \to N \otimes \mathbb{R}.$$

Let M' and N' be the dual lattices of M and N. We have the following inclusions

$$M \subset L,\ N \subset L,\ M \oplus N \subseteq L \subseteq L' \subseteq M' \oplus N',$$

and equalities of the sets

$$\mathrm{pr}_M(L') = M',\ \mathrm{pr}_N(L') = N'.$$

Consider a rectangular $|L'/L| \times |N'/N|$ dimensional matrix $T_{L,N}$ with entries

$$\vartheta_{\lambda,\nu}(\tau) = \sum_{\substack{m \in M': \\ m+\nu \in \lambda + L}} \mathbf{e}\big(-\mathrm{q}(m)\tau\big)$$

where $\lambda \in L'/L, \nu \in N'/N, \tau \in \mathfrak{H}$. This sum is well defined since $N \subset L$. Note that the lattice M is negative definite and hence the series converge.

Theorem 3. *Suppose that the lattices L, M and N are defined as above. Then there is a map $T_{L,N} : M_k(\rho_L) \to M_{k+d/2}(\rho_N)$ sending a function $f = (f_\lambda)_{\lambda \in L'/L}$ to $g = (g_\nu)_{\nu \in N'/N}$ defined as*

$$g_\nu(\tau) = \sum_{\lambda \in L'/L} \vartheta_{\lambda,\nu}(\tau)\, f_\lambda(\tau). \tag{16}$$

In other words

$$g = T_{L,N} f$$

where f and g are considered as column vectors.

Proof. Consider the function

$$\Theta_{M(-1)}(\tau) = \overline{\Theta_M(\tau)} = \sum_{\mu \in M'/M} e_\mu \sum_{m \in M+\mu} \mathbf{e}(-\mathrm{q}(m)\tau)$$

that belongs to $M_{d/2}(\rho_{M(-1)})$. It follows from (16) and (12) that

$$T_{L,N}(f) = \left\langle \mathrm{res}_{L/M\oplus N}(f), \Theta_{M(-1)} \right\rangle_{\mathbb{C}[M'/M]}.$$

Thus, from Lemma 2 we deduce that $T_{L,N}(f)$ is in $M_{k+d/2}(\rho_N)$. □

Theorem 4. *Let L, M, N be as above. Denote by $i : \mathrm{Gr}^+(N) \to \mathrm{Gr}^+(L)$ a natural embedding induced by inclusion $N \subset L$. Then, for $v^+ \in \mathrm{Gr}^+(N)$ the theta lift of a function $f \in M^!_{1-b/2}(\rho_L)$ the following holds*

$$\Phi_L(i(v^+), f) = \Phi_N(v^+, T_{L,N}(f)). \tag{17}$$

Proof. For a vector $\ell \in L'$ denote $m = \mathrm{pr}_M(\ell)$ and $n = \mathrm{pr}_N(\ell)$. Recall that $m \in M'$ and $n \in N'$. Since v^+ is an element of $\mathrm{Gr}^+(N)$ it is orthogonal to M. We have

$$\mathrm{q}(\ell_{v^+}) = \mathrm{q}(n_{v^+}),\ \mathrm{q}(\ell_{v^-}) = \mathrm{q}(m) + \mathrm{q}(n_{v^-}).$$

Thus for $\lambda \in L'/L$ we obtain

$$\Theta_{\lambda+L}(\tau, v^+) = \sum_{\ell \in \lambda+L} \mathbf{e}\big(\mathrm{q}(\ell_{v^+})\tau + \mathrm{q}(\ell_{v^-})\bar{\tau}\big) =$$

$$\sum_{\substack{m \in M',\, n \in N': \\ m+n \in \lambda+L}} \mathbf{e}\big(\mathrm{q}(n_{v^+})\tau + \mathrm{q}(n_{v^-})\bar{\tau} + \mathrm{q}(m)\bar{\tau}\big).$$

Since $N \subset L$ we can rewrite this sum as

$$\Theta_{\lambda+L}(\tau, v^+) = \sum_{\nu \in N'/N} \Theta_{\nu+N}(\tau, v^+)\, \overline{\vartheta_{\nu,\lambda}(\tau)}.$$

Thus, we see that for $f = (f_\lambda)_{\lambda \in L'/L}$ the following scalar products are equal

$$\langle f, \overline{\Theta_L(\tau, v^+)} \rangle = \langle T_{L,N}(f), \overline{\Theta_N(\tau, v^+)} \rangle.$$

So, the regularized integrals (15) of both sides of the equality are also equal. □

5 Local and global heights on curves

In this section we review the basic ideas of Néron's theory. A more detailed overview of this topic is given in [11]. Let X be a non-singular, complete, geometrically connected curve over the locally compact field k. We normalize the valuation map $|\ |_v : k_v \to \mathbb{R}_+^\times$ so that for any Haar measure dx on k_v we have the formula $\alpha^*(dx) = |\alpha|_v \cdot dx$.

Let a and b denote divisors of degree zero on X over k_v with disjoint support. Then Néron defines a *local symbol* $\langle a, b\rangle_v$ in $\mathbb{R}$ which is

(i) bi-additive,

(ii) symmetric,

(iii) continuous,

(vi) satisfies the property $\langle \sum m_x(x), (f)\rangle_v = \log |\prod f(x)^{m_x}|_v$, when $b = (f)$ is principal.

These properties characterize the local symbol completely.

When v is archimedean, one can compute the Néron symbol as follows. Associated to b is a Green's function G_b on the Riemann surface $X(\overline{k_v}) - |b|$ which satisfies $\partial\overline{\partial} G_b = 0$ and has logarithmic singularities at the points in $|b|$. More precisely, the function $G_b - \mathrm{ord}_z(b) \log |\pi|_v$, is regular at every point z, where π is a uniformizing parameter at z. These conditions characterize G_b up to the addition of a constant, as the difference of any two such functions would be globally harmonic. The local formula for $a = \sum m_x(x)$ is then

$$(a, b)_v = \sum m_x\, G_b(x).$$

This is well-defined since $\sum m_x = 0$ and satisfies the required properties since if $b = (f)$ we could take $G_b = \log |f|$.

If v is a non-archimedean place, let $\mathfrak{o}_v$ denote the valuation ring of k_v and q_v the cardinality of the residue field. Let $\mathcal{X}$ be a regular model for X over $\mathfrak{o}_v$ and extend the divisors a and b to divisors A and B of degree zero on $\mathcal{X}$. These extensions are not unique, but if we insist that A have zero intersection with each fibral component of $\mathcal{X}$ over the residue field, then the intersection product $(A \cdot B)$ is well defined. We have the formula

$$\langle a, b\rangle_v = -(A \cdot B) \log q_v.$$

Finally, if X, a, and b are defined over the global field k we have $(a,b)_v = 0$ for almost all completions k_v and the sum

(18) $$\langle a, b\rangle = \sum_v \langle a, b\rangle_v$$

depends only on the classes of a and b in the Jacobian. This is equal to the global height pairing of Néron and Tate.

It is desirable to have an extension of the local pairing to divisors a and b of degree 0 on X which are not relatively prime. At the loss of some functoriality, this is done in [11] as follows.

At each point x in the common support, choose a basis $\frac{\partial}{\partial t}$ for the tangent space and let π be a uniformizing parameter with $\frac{\partial \pi}{\partial t} = 1$. Any function $f \in k_v(X)^*$ then has a well-defined "value" at x:

$$f[x] = \frac{f}{z^m}(x) \text{ in } k_v^*,$$

where $m = \mathrm{ord}_x f$. This depends only on $\frac{\partial}{\partial t}$, not on π. Clearly we have

$$fg[x] = f[x]\, g[x].$$

To pair a with b we may find a function f on X such that $b = \mathrm{div}(f) + b'$, where b' is relatively prime to a. We then define

(19) $$\langle a, b\rangle_v = \log|f[a]|_v + \langle a, b'\rangle.$$

This definition is independent of the choice of f used to move b away from a. The same decomposition formula (18) into local symbols can be used even when the divisors a and b have a common support provided that the uniformizing parameter π at each point of their common support is chosen over k.

6 Embedding argument

In this section for each $\mathfrak{b} \in \mathrm{CL}_K$ and each $f \in S_1^!(\rho)$ we construct a meromorphic function Ψ on the modular curve $X_0(D)$ that satisfies the following two properties: this function has zeroes and poles at the Heegner points; the identity

(20) $$(f, \Theta_{\mathfrak{b}})^{\mathrm{reg}} = \log|\Psi(\mathfrak{z})|$$

holds for some CM-point $\mathfrak{z}$. Our main tools would be Borcherds lifts and see-saw identities introduced in Section 3.

First, let us introduce a convenient lattice that contains the fractional ideal $\mathfrak{b}$ as a sublattice. Consider the lattice

$$L = \left\{ \begin{pmatrix} A/D & B \\ B & C \end{pmatrix} \Big| A, B, C \in \mathbb{Z} \right\} \tag{21}$$

equipped with the quadratic form $\mathrm{q}(x) := -D \det(x)$.

For $\ell \in L'$ with $\mathrm{q}(\ell) < 0$ denote by $\mathfrak{z}_\ell$ the point in $\mathfrak{H}$ corresponding to the positive definite subspace $\ell^\perp$ via (29). More explicitly, for the vector

$$\ell = \begin{pmatrix} \gamma & -\beta/2 \\ -\beta/2 & \alpha \end{pmatrix}$$

the point $\mathfrak{z}_\ell$ satisfies the quadratic equation

$$\alpha \mathfrak{z}_\ell^2 + \beta \mathfrak{z}_\ell + \gamma = 0. \tag{22}$$

The following lemma is crucial for the construction of the meromorhpic function Ψ that satisfies (20).

Lemma 3. *For $D > 0$ and $D \equiv 0$ or $3 \bmod 4$ consider the lattice L defined by (21). Let $m \in L$ be a vector of norm -1 and denote $N = L \cap m^\perp$. Denote by $\mathfrak{c}$ the fractional ideal $\mathfrak{z}_m\mathbb{Z} + \mathbb{Z}$. Then the following holds*
(i) the lattice N is isomorphic to the fractional ideal $\mathfrak{c}^2$ equipped with the quadratic form $\mathrm{q}(\gamma) = \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{c}^2)} N_{K/\mathbb{Q}}(\gamma)$.
(ii) $L = N \oplus \mathbb{Z}m$.

Proof. First we prove part (i). Suppose that

$$m = \frac{1}{D} \begin{pmatrix} c & -b/2 \\ -b/2 & a \end{pmatrix}$$

for some $a, b, c \in \mathbb{Z}$. Denote

$$Z = \frac{a}{\sqrt{-D}} \begin{pmatrix} \mathfrak{z}_m^2 & \mathfrak{z}_m \\ \mathfrak{z}_m & 1 \end{pmatrix}.$$

Consider the map

$$\imath : K \to L \otimes \mathbb{Q}$$

defined by

$$s \to sZ + \overline{s}\overline{Z}.$$

It maps K to $m^{\perp}$ and is an isometry, assuming that the quadratic form on K is given by $\mathrm{q}(\beta) = N_{K/\mathbb{Q}}(\beta)$ and the quadratic form on $L \otimes \mathbb{Q}$ is given by $\mathrm{q}(\ell) = -\det(\ell)$. We have

$$\begin{aligned} \imath(1) &= \frac{1}{a}\begin{pmatrix} b & -a \\ -a & 0 \end{pmatrix}, \\ \imath(\mathfrak{z}_m) &= \frac{1}{a}\begin{pmatrix} -c & 0 \\ 0 & a \end{pmatrix}, \\ \imath(\mathfrak{z}_m^2) &= \frac{1}{a}\begin{pmatrix} 0 & c \\ c & -b \end{pmatrix}. \end{aligned}$$

Thus, we find

$$S_2(\mathbb{Z}) \cap \begin{pmatrix} c & -b/2 \\ -b/2 & a \end{pmatrix}^{\perp} = a\,\imath(1)\mathbb{Z} + a\,\imath(\mathfrak{z}_m)\mathbb{Z} + a\,\imath(\mathfrak{z}_m^2)\mathbb{Z}. \tag{23}$$

On the other hand

$$\mathfrak{c}^2 = \mathbb{Z} + \mathfrak{z}_m\mathbb{Z} + \mathfrak{z}_m^2\mathbb{Z}. \tag{24}$$

The quadratic form on $\mathfrak{c}^2$ is given by $\mathrm{q}(\gamma) = a^2 N_{K/\mathbb{Q}}(\gamma)$ for $\gamma \in \mathfrak{c}^2$. Hence, we check that

$$\begin{aligned} \mathrm{q}(1) &= \mathrm{q}(a\,\imath(1)), \\ \mathrm{q}(\mathfrak{z}_m) &= \mathrm{q}(a\,\imath(\mathfrak{z}_m)), \\ \mathrm{q}(\mathfrak{z}_m^2) &= \mathrm{q}(a\,\imath(\mathfrak{z}_m^2)). \end{aligned}$$

Now the part (i) of the lemma follows from the equations (23) and (24).

Now we prove (ii). Denote $M := m\mathbb{Z}$. We have the following inclusions

$$M' \oplus N' \subseteq L' \subset L \subseteq M \oplus N.$$

Observe that

$$|L'/L| = 2D, \quad |M'/M| = 2, \quad |N'/N| = D.$$

Thus, $L = M \oplus N$ and $L' = M' \oplus N'$. $\square$

Our next goal is to find a preimage of a function $f \in M_1^!(\rho_N)$ under the map $T_{L,N}$ defined in Theorem 4.

Theorem 5. *Let the lattices L, N, and the vector m be as in Lemma 3. Suppose that $f \in M_1^!(\rho_N)$ is a modular form with zero constant term and rational Fourier coefficients. Then there exists a function $h \in S_{1/2}^!(\rho_L)$ such that:*

(i) the function $h(\tau) = \sum_{\beta \in L'/L} e_\beta \sum_{d \in \mathbb{Z}} b_\beta(d)\, \mathbf{e}(\frac{d}{4D}\tau)$ has rational Fourier coefficients

(ii) the Fourier coefficients of h satisfy $b_\beta(-Ds^2) = 0$ for all $s \in \mathbb{Z}$ and $\beta \in L'/L$,

(iii) $T_{L,N}(h) = f$.

Proof. Denote by S the lattice $\mathbb{Z}$ equipped with the quadratic form $\mathrm{q}(x) := -x^2$. For this lattice we have $S'/S \cong \mathbb{Z}/2\mathbb{Z}$. Lemma 3 implies that $L \cong N \oplus S$. Note that $L'/L \cong S'/S \times N'/N$ and $\rho_L = \rho_S \otimes \rho_N$. Denote

$$\theta_0(\tau, z) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau + 2nz), \quad \theta_1(\tau, z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \mathbf{e}(n^2\tau + 2nz)$$

and

$$\theta_\kappa(\tau) = \theta_\kappa(\tau, 0), \quad \kappa = 0, 1.$$

It follows from the definition of $T_{L,N}$ that

$$(T_{L,N}(h))_\nu = \sum_{\kappa \in S'/S} h_{(\kappa,\nu)} \theta_\kappa.$$

Let $\tilde{\phi}_{-2,1}$, $\tilde{\phi}_{0,1}$ be the weak Jacobi forms defined in the book [9] p.108. These functions can be written as

$$\tilde{\phi}_{-2,1}(\tau, z) = \psi_0(\tau)\, \theta_0(\tau, z) + \psi_1(\tau)\, \theta_1(\tau, z),$$
$$\tilde{\phi}_{0,1}(\tau, z) = \varphi_0(\tau)\, \theta_0(\tau, z) + \varphi_1(\tau)\, \theta_1(\tau, z)$$

where

$$\psi_0 = -2 - 12q - 56q^2 - 208q^3 + \cdots, \qquad (25)$$
$$\psi_1 = q^{-1/4} + 8q^{3/4} + 39q^{7/4} + 152q^{11/4} + \cdots$$

$$\varphi_0 = 10 + 108q + 808q^2 + 4016q^3 + \cdots,$$
$$\varphi_1 = q^{-1/4} - 64q^{3/4} - 513q^{7/4} - 2752q^{11/4} + \cdots.$$

The vector valued functions (ψ_0, ψ_1) and (φ_0, φ_1) belong to spaces $M^!_{-5/2}(\rho_S)$ and $M^!_{-1/2}(\rho_S)$ respectively, and they satisfy

$$\begin{aligned} \tilde{\phi}_{-2,1}(\tau,0) &= \psi_0(\tau)\,\theta_0(\tau) + \psi_1(\tau)\,\theta_1(\tau) = 0, \\ \tilde{\phi}_{0,1}(\tau,0) &= \varphi_0(\tau)\,\theta_0(\tau) + \varphi_1(\tau)\,\theta_1(\tau) = 12. \end{aligned} \tag{26}$$

First, we construct a function $g \in M^!_{1/2}(\rho_L)$ that satisfies conditions (i) and (iii). Define

$$g_{(\kappa,\nu)} := \frac{1}{12}\varphi_\kappa f_\nu - \frac{c}{12}\psi_\kappa \tilde{f}_\nu,\ (\kappa,\nu) \in S'/S \times N'/N. \tag{27}$$

This function satisfies

$$\begin{aligned} T_{L,N}(g) =& \frac{1}{12}\sum_{\nu\in N'/N} e_\nu\,(g_{(0,\nu)}\theta_0 + g_{(1,\nu)}\theta_1) \\ =& \frac{1}{12}\sum_{\nu\in N'/N} e_\nu\, f_\nu(\varphi_0\theta_0 + \varphi_1\theta_1) \\ =& f. \end{aligned}$$

Next, we will add a correction term to g and construct a function that satisfies also (ii). Function g has the Fourier expansion of the form

$$g(\tau) = \sum_{\beta\in L'/L} e_\beta \sum_{d\in\mathbb{Z}} a_\beta(d)\,\mathbf{e}(\frac{d}{4D}\tau).$$

Let s be the minimal integral number such that $a(-Ds^2) \neq 0$. Choose a supplementary function $\tilde{g} \in M^!_{1/2}(\rho_L)$ defined as

$$\tilde{g}_{(\kappa,\nu)} := \psi_\kappa\,[\,j^{\frac{s^2-t}{4}}, \Theta_{N+\nu}\,],\ (\kappa,\nu) \in S'/S \times N'/N, \tag{28}$$

where

$$t = \begin{cases} 0 & \text{if } s \equiv 0 \bmod 2 \\ 1 & \text{otherwise} \end{cases}$$

and $[\,\cdot\,,\,\cdot\,]$ denotes the Rankin-Cohen brackets. This function satisfies

$$\begin{aligned} T_{L,N}(\tilde{g}) &= \sum_{\nu\in N'/N} e_\nu\,(\tilde{g}_{(0,\nu)}\theta_0 + \tilde{g}_{(1,\nu)}\theta_1) \\ &= \sum_{\nu\in N'/N} e_\nu\, f_\nu(\psi_0\theta_0 + \psi_1\theta_1) \\ &= 0. \end{aligned}$$

Suppose that the function $\tilde{g}$ has the Fourier expansion

$$g(\tau) = \sum_{\beta \in L'/L} e_\beta \sum_{d \in \mathbb{Z}} \tilde{a}_\beta(d)\, \mathbf{e}(\frac{d}{4D}\tau).$$

Consider the function

$$\tilde{h} = g - \frac{a_{\kappa,0}(-Ds^2)}{\tilde{a}_{\kappa,0}(-Ds^2)}\tilde{g} \in M^!_{1/2}(\rho_L).$$

This function has the Fourier expansion

$$\tilde{h}(\tau) = \sum_{\beta \in L'/L} e_\beta \sum_{d \in \mathbb{Z}} \tilde{b}_\beta(d)\, \mathbf{e}(\frac{d}{4D}\tau).$$

Denote by $\tilde{s}$ be the minimal integral number such that $\tilde{b}(-Ds^2) \neq 0$. Suppose that $s > 0$, then $\tilde{s} < s$. Hence, repeating this procedure we can construct a function $h \in M^!_{1/2}(\rho_L)$ such that

$$h(\tau) = \sum_{\beta \in L'/L} e_\beta \sum_{d \in \mathbb{Z}} b_\beta(d)\, \mathbf{e}(\frac{d}{4D}\tau)$$

has rational Fourier coefficients, the Fourier coefficients of h satisfy $b_\beta(-Ds^2) = 0$ for all $s \geq 1$ and $\beta \in L'/L$, and

$$T_{L,N}(h) = f.$$

In particular, we have

$$h_{0,0}\theta_0 + h_{1,0}\theta_1 = f_0.$$

Hence, the constant terms of these functions are equal

$$\mathrm{CT}(h_{0,0}\theta_0 + h_{1,0}\theta_1) = \mathrm{CT}(f_0) = 0.$$

On the other hand

$$\mathrm{CT}(h_{0,0}\theta_0 + h_{1,0}\theta_1) = \sum_{s \in \mathbb{Z}} b(-Ds^2) = b(0).$$

Thus, the function h satisfies the conditions (i)-(iii) of the theorem. This finishes the proof. $\square$

We observe that the Grassmanian $\mathrm{Gr}^+(L)$ is isomorphic to the upper half-plane $\mathfrak{H}$. There is a map $\mathfrak{H} \to \mathrm{Gr}^+(L)$ given by

$$z \quad \to \quad v^+(z) := \Re \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \mathbb{R} + \Im \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix} \mathbb{R} \subset L \otimes \mathbb{R}. \tag{29}$$

The group $\Gamma_0(D)$ acts on L' and fixes all the elements of L'/L. Denote by $X_0(D)$ the modular curve $\overline{\Gamma_0(D)\backslash\mathfrak{H}}$.

Suppose that the vector $m \in L$, The lattice N and the point $\mathfrak{z}_m \in \mathfrak{H}$ are defined as in Theorem 5. Let h be the modular form $h \in S^!_{1/2}(\rho_L)$ satisfying

$$T_{L,N}(h) = f, \tag{30}$$

which was constructed in the previous theorem. It follows from (30) and Theorem 4 that

$$\Phi_L(h, \mathfrak{z}_\mathfrak{m}) = \Phi_N(f).$$

Recall that by definition

$$\Phi_N(f) = (f, \Theta_{\mathfrak{c}^2})^{\mathrm{reg}}.$$

Without loss of generality assume that h has integral negative Fourier coefficients. The infinite product $\Psi(z) := \Psi_L(h, z)$ introduced in Section 3 defines a meromorphic function on $X_0(D)$ with only zeroes and poles at Heegner points. Theorem 2 in Section 3 implies

$$(f, \Theta_{\mathfrak{c}^2})^{\mathrm{reg}} = \log |\Psi_L(h, \mathfrak{z}_\mathfrak{m})|. \tag{31}$$

It also follows from Theorem 2 that the divisor of Ψ_L is supported at Heegner points.

7 Heights of Heegner points

In this section we compute the local height pairing between Heegner divisors. These calculations are carried out in the celebrated series of papers [12], [13]. For the convenience of the reader we recall the main steps of the computation in what follows.

First, let as recall the definition of Heegner points and the way they can be indexed by the vectors of the lattice L'.

For $\ell \in L'$ with $\mathrm{q}(\ell) < 0$ denote by x_ℓ the divisor $(\mathfrak{z}_\ell) - (\infty)$ on the modular curve $X_0(D)$. The divisor x_ℓ is defined over the Hilbert class field of $\mathbb{Q}(\sqrt{D\mathrm{q}(\ell)})$.

For any integer $d > 0$ such that $-d$ is congruent to a square modulo $4D$, choose a residue $\beta (\bmod 2D)$ with $-d \equiv \beta^2 (\bmod 4D)$ and consider the set

$$L_{d,\beta} = \left\{ \ell = \begin{pmatrix} a/D & b/2D \\ b/2D & c \end{pmatrix} \in L' \;\middle|\; \mathrm{q}(\ell) = -\frac{d}{4D}, b \equiv \beta (\bmod 2D) \right\}$$

on which $\Gamma_0(D)$ acts. Define the Heegner divisor

$$\mathrm{y}_{d,\beta} = \sum_{\ell \in \Gamma_0(D) \backslash L_{d,\beta}} \mathrm{x}_\ell.$$

The Fricke involution acts on L' by

$$\ell \to \frac{1}{D} \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix} \ell \begin{pmatrix} 0 & -D \\ 1 & 0 \end{pmatrix}$$

and maps $L_{d,\beta}$ to $L_{d,-\beta}$. Denote

$$\mathrm{y}^*_d = \mathrm{y}_{d,\beta} + \mathrm{y}_{d,-\beta}. \tag{32}$$

The divisor y^*_d is defined over $\mathbb{Q}$ ([13] p. 499.)

Now we would like to compute the local height pairings between the divisor x_ℓ and a Heegner divisor. The definition of the local height pairing is given in Section 5. The following theorem can be deduced from the computations in Section IV.4 in [13].

Theorem 6. *Let $d_1,\ d_2 > 0$ be two integers and $\beta_1,\ \beta_2$ be two elements of $\mathbb{Z}/4D\mathbb{Z}$ with $-d_1 \equiv \beta_1^2 (\bmod 4D)$ and $-d_2 \equiv \beta_2^2 (\bmod 4D)$. Suppose that d_1 is fundamental and d_2/d_1 is not a full square. Fix a vector $\ell \in L_{d_1,\beta_1}$. Let p be a prime with $\left(\frac{p}{d_1}\right) = -1$ and $\gcd(p, D) = 1$. Choose a prime ideal $\wp$ lying above p in the Hilbert class field of $\mathbb{Q}(\sqrt{-d_1})$. Then the following formula for the local height holds*

$$\langle \mathrm{x}_\ell, \mathrm{y}^*_{d_2} \rangle_\wp = \log(p) \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \beta_1\beta_2 \bmod 2}} \delta_{d_1}(r)\, r_{\bar{\mathfrak{c}}^2\mathfrak{a}^2}\left(\frac{d_1 d_2 - r^2}{4Dp}\right) \mathrm{ord}_p\left(\frac{d_1 d_2 - r^2}{4D}\right),$$

where $\mathfrak{c} = \mathbb{Z}\mathfrak{z}_\ell + \mathbb{Z}$, $\mathfrak{n} = \mathbb{Z}D + \mathbb{Z}\frac{\beta_1+\sqrt{-d_1}}{2}$, *and the ideal* $\mathfrak{a}$ *is defined by* (6). *Here*

$$\delta_d(r) = \begin{cases} 2 & \textit{for } r \equiv 0 \bmod d \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. The curve $X_0(D)$ may be described over $\mathbb{Q}$ as the compactification of the space of moduli of elliptic curves with a cyclic subgroup of order D [12]. Over a field k of characteristic zero, the points y of $X_0(D)$ correspond to diagrams

$$\psi : F \to F',$$

where F and F' are (generalized) elliptic curves over k and ψ is an isogeny over k whose kernel is isomorphic to $\mathbb{Z}/D\mathbb{Z}$ over an algebraic closure $\overline{k}$.

Point $\mathfrak{z}_\ell \in \mathfrak{H}$ defines the point $\mathrm{x} \in X_0(D)$. Then $\mathrm{x} = (\phi : E \to E')$ and over $\mathbb{C}$ this diagram is isomorphic to

$$\mathbb{C}/\mathfrak{c} \xrightarrow{\mathrm{id}_{\mathbb{C}}} \mathbb{C}/\mathfrak{cn}\ .$$

Following the calculations in [12] we reduce the computation of local heights to a problem in arithmetic intersection theory. Let us set up some notations. Denote by v the place of H_{d_1}, the Hilbert class field of $\mathbb{Q}(\sqrt{-d_1})$, corresponding to prime ideal $\wp$. Denote by Λ_v the ring of integers in the completion $H_{d_1,v}$ and let π be an uniformizing parameter in Λ_v. Let W be the completion of the maximal unramified extension extension of Λ_v. Let $\underline{X}$ be a regular model for X over Λ_v and $\underline{\mathrm{x}}$, $\underline{\mathrm{y}}$ be the sections of $\underline{X} \otimes \Lambda_v$ corresponding to the points x and y. A model that has a modular interpretation is described in [12] Section III.3). The general theory of local height pairing [11] implies

$$\langle \mathrm{x}, \mathrm{y} \rangle_v = -(\underline{\mathrm{x}} \cdot \underline{\mathrm{y}}) \log p.$$

The intersection product is unchanged if we extend scalars to W. By Proposition 6.1 in [12]

$$(\underline{\mathrm{x}} \cdot \underline{\mathrm{y}})_W = \frac{1}{2} \sum_{n \geq 1} \mathrm{CardHom}_{W/\pi^n}(\underline{\mathrm{x}}, \underline{\mathrm{y}})_{\mathrm{deg}1}.$$

Denote by R the ring $\mathrm{Hom}_{W/\pi}(\underline{\mathrm{x}_\ell})$. On p. 550 of [13] the following formula for the intersection number is obtained via the lifting theorem (Proposition

2.7 of [10])

$$(\underline{\mathrm{x}_\ell} \cdot \underline{\mathrm{y}^*_{d_2}})_W = \frac{1}{4} \sum_{\substack{r^2 < d_1 d_2 \\ r \equiv \beta_1\beta_2 \pmod{2D}}} \mathrm{Card}\big\{S_{[d_1,2r,d_2]} \to R \bmod R^\times\big\}\, \mathrm{ord}_p\left(\frac{d_1 d_2 - r^2}{4D}\right),$$

where $S_{[d_1,2r,d_2]}$ is the Clifford order

$$S_{[d_1,2r,d_2]} = \mathbb{Z} + \mathbb{Z}\frac{1+e_1}{2} + \mathbb{Z}\frac{1+e_2}{2} + \mathbb{Z}\frac{(1+e_1)(1+e_2)}{4},$$

$$e_1^2 = -d_1, \quad e_2^2 = -d_2, \quad e_1e_2 + e_2e_1 = 2r.$$

Formula (9.3) in [12] gives us a convenient description of the ring R. Namely, for $a, b \in \mathbb{Q}(\sqrt{-d_1})$ denote

$$[a,b] = \begin{pmatrix} a & b \\ p\bar{b} & \bar{a} \end{pmatrix}$$

and consider the quaternion algebra over $\mathbb{Q}$

$$B = \left\{ [a,b] \,\middle|\, a, b \in \mathbb{Q}(\sqrt{-d_1}) \right\}.$$

Then R is an Eichler order of index D in this quaternion algebra and it is given by

$$R = \left\{ [a,b] \,\middle|\, a \in \mathfrak{d}^{-1},\ b \in \mathfrak{d}^{-1}\mathfrak{n}\overline{\mathfrak{a}\mathfrak{c}}\mathfrak{a}^{-1}\mathfrak{c}^{-1}, a \equiv b \bmod \mathfrak{o}_{d_1} \right\},$$

where $\mathfrak{d}$ is the different of $\mathbb{Q}(\sqrt{-d_1})$.

By the same computations as in Lemma 3.5 of [10] we find that the number of embeddings of $S_{[d_1,2r,d_2]}$ into R, normalized so that the image of e_1 is $[\sqrt{-d_1}, 0]$, is equal to

$$\delta_{d_1}(r)\, r_{\bar{\mathfrak{c}}^2\mathfrak{a}^2}\left(\frac{d_1d_2 - r^2}{4Dp}\right) \mathrm{ord}_p\left(\frac{d_1d_2 - r^2}{4D}\right).$$

This finishes the proof of the theorem. $\square$

8 Proof of Theorem 1.

Proof of Theorem 1. Since the discriminant $-D$ is prime, the class number of K is odd and there exists an ideal $\mathfrak{c}$ such that $\mathfrak{b} \sim \overline{\mathfrak{c}}^2$ in the ideal class group. Suppose that

$$\mathfrak{c} = \mathfrak{z}\mathbb{Z} + \mathbb{Z}, \tag{33}$$

where $\mathfrak{z}$ is a CM point of discriminant $-D$. Property (33) is preserved when we act on $\mathfrak{z}$ by elements of $\mathrm{SL}_2(\mathbb{Z})$. Thus, we may assume that $\mathfrak{z}$ satisfies the quadratic equation

$$aD\mathfrak{z}^2 + bD\mathfrak{z} + c = 0$$

for $a, b, c \in \mathbb{Z}$ and $b^2D^2 - 4Dac = -D$. The matrix

$$m = \begin{pmatrix} 2c/D & -b \\ -b & 2a \end{pmatrix}$$

belongs to the lattice L and has the norm -1. Lemma 3 implies that the lattice $N := L \cap m^{\perp}$ corresponds to the fractional ideal $\mathfrak{c}^2$ as explained in Section 3 and moreover, the lattice L splits as $L = N \oplus m\mathbb{Z}$.

Next, by Theorem 5 we find a weak cusp form $h \in S^!_{1/2}(\rho_L)$ satisfying

$$T_{L,N}(h) = f, \tag{34}$$

where $T_{L,N}$ is defined as in Theorem 4. Function h has the Fourier expansion of the form

$$h(\tau) = \sum_{\beta \in \mathbb{Z}/2D\mathbb{Z}} e_\beta \sum_{d \equiv \beta^2 \bmod 4D} b(d)\, \mathbf{e}\Big(\frac{d}{4D}\tau\Big).$$

It follows from (34) and Theorem 4 that

$$\Phi_N(f) = \Phi_L(h, \mathfrak{z}).$$

From Theorem 2 in Section 3 we know that

$$\Phi_L(h, \mathfrak{z}) = \log|\Psi_L(h, \mathfrak{z})|, \tag{35}$$

where $\Psi(z) = \Psi_L(h, z)$ is a meromorphic function. Theorem 2 also implies that

$$\mathrm{div}(\Psi) = \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^*, \tag{36}$$

where y_d^* is the Heegner divisor defined in (32).

Denote $\mathrm{x} = (\mathfrak{z}) - (\infty)$. The condition (ii) of Theorem 5 implies that the function $\Phi_L(h, \cdot)$ is real analytic at point $\mathfrak{z}$. Thus, the only point at the common support of x and $\mathrm{div}(\Psi)$ is ∞. In order to compute height pairing between x and $\mathrm{div}(\Psi)$ we must fix a uniformizing parameter π at this cusp. We let π denote the Tate parameter q on the family of degenerating elliptic curves near ∞. This is defined over $\mathbb{Q}$. Over $\mathbb{C}$ we have $q = \mathbf{e}(z)$ on $X_0^*(D) = \Gamma_0^* \backslash \overline{\mathfrak{H}}$, where $z \in \mathfrak{H}$ with $\Im(z)$ sufficiently large.

Recall that the divisors x and $\mathrm{div}(\Psi)$ are defined over H. The axioms of local height (listed in Section 5) together with the refined definition (19) imply that for each prime $\wp$ of H

$$\mathrm{ord}_\wp\big(\Psi(\mathfrak{z})\big)\log p - \mathrm{ord}_\wp\big(\Psi[\infty]\big)\log p = \Big\langle \mathrm{x}, \sum_{d=1}^{\infty} b(-d)\, \mathrm{y}_d^* \Big\rangle_\wp. \tag{37}$$

From the infinite product of Theorem 13.3 in [2] we find that $\Psi[\infty] = 1$ for the choice of the uniformizing parameter at ∞ as above. Theorem 5 part (ii) implies that d/D is not a full square provided $b(-d) \neq 0$. Thus, by Theorem 6 we obtain

$$\langle \mathrm{x}, \mathrm{y}_d^* \rangle_\wp = \log(p) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv d (\mathrm{mod}\ 2)}} r_{\bar{\mathfrak{c}}^2 \mathfrak{a}^2}\left(\frac{d - Dn^2}{4p}\right) \mathrm{ord}_p\left(\frac{d - Dn^2}{4}\right). \tag{38}$$

Note that the sum

$$\sum_{d=0}^{\infty} b(-d) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv d (\mathrm{mod}\ 2)}} r_{\bar{\mathfrak{c}}^2 \mathfrak{a}^2}\left(\frac{d - Dn^2}{4p}\right) \mathrm{ord}_p\left(\frac{d - Dn^2}{4}\right)$$

is equal to the constant term with respect to $\mathbf{e}(\tau)$ of the following series

$$\sum_{\nu \in \mathbb{Z}/D\mathbb{Z}} \left(\Big(h_{0,\nu}\theta_0 + h_{1,\nu}\theta_0 \Big) \sum_{t \equiv \nu \bmod D} r_{\mathfrak{b}\mathfrak{a}^2}\left(\frac{t}{p}\right) \mathrm{ord}_p(t)\, \mathbf{e}\Big(\frac{t}{D}\tau\Big) \right).$$

The equation (34) implies

$$f_\nu = h_{(0,\nu)}\theta_0 + h_{(1,\nu)}\theta_1, \quad \nu \in \mathbb{Z}/D\mathbb{Z}, (\kappa, \nu) \in \mathbb{Z}/2D\mathbb{Z}. \tag{39}$$

Hence, combining the equations (38) and (39) we arrive at

$$\Big\langle \mathrm{x}, \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^* \Big\rangle_{\wp} = \log p \sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} \sum_{t=0}^{\infty} c_\nu(-t)\, r_{\mathfrak{ba}^2}\Big(\frac{t}{p}\Big) \operatorname{ord}_p(t).$$

Finally, the equations (35) and (37) imply

$$\operatorname{ord}_{\wp}(\alpha) = \operatorname{ord}_{\wp}(\Psi_L(h,\mathfrak{z})) = \frac{1}{\log p} \Big\langle \mathrm{x}, \sum_{d=0}^{\infty} b(-d)\, \mathrm{y}_d^* \Big\rangle_{\wp} =$$

$$= \sum_{\nu\in\mathbb{Z}/D\mathbb{Z}} \sum_{t=0}^{\infty} c_\nu(-t)\, r_{\mathfrak{ba}^2}\Big(\frac{t}{p}\Big) \operatorname{ord}_p(t).$$

This finishes the proof of Theorem 1. $\square$

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